GREEN'S FUNCTION OF STEADY-HEAT-CONDUCTION OPERATOR FOR SOME SHELLS OF REVOLUTION

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A method is outlined for obtaining accurate representations of Green's function of the steady-heat-conduction operator for thin spherical and toroidal shells.

Although the boundary problems of steady heat conduction are among the simplest in mathematical physics, their solution is associated with considerable difficulties in many specific cases [1]. Bearing in mind that in applied problems, especially in optical and radioelectronic engineering [2, 3], higher demands are in practice often imposed on the accuracy in determining the temperature field of the elements and units under consideration, the growing tendency among investigators to develop universal and high-accuracy algorithms for the solution of these problems is quite understandable.

Of the boundary problems of steady heat conduction, problems of determining the temperature fields in regions circumscribed by thin shells of revolution form one typical class that is nontrivial from the viewpoint of the computational algorithm [4, 5]. The difficulties arise because it is not often possible to find a satisfactory parameterization of the median surface of the given shell such that the Laplacian can be written in a sufficiently simple form. Usually the Laplacian takes the form of an operator with variable coefficients, and as a result it is difficult to use many of the known methods [6, 8] which are useful for the solution of simpler problems.

Often the situation is further complicated because it is necessary to formulate and solve the problem for a region whose boundaries cannot be described by coordinate lines in the chosen coordinate system and this leads to considerable computational difficulties even in the case of a Cartesian Laplacian [9].

These two circumstances create a situation which seems particularly to favor the use of an algorithm based on Green's function of the considered operator constructed for a region whose boundaries partially coincide with the boundaries of the given body [1, 9, 10].

In such problems, generally speaking, no difficulties attach to the construction of approximate representations of Green's functions by the methods used, in particular, in [1, 9, 10]. However, in many cases of practical importance it is even possible to obtain accurate expressions for Green's functions. The present paper describes a method of constructing Green's functions for a spherical closed shell, a spherical segment, and a closed toroidal shell with a circular meridian. By using these representations of Green's functions, it is possible to construct an effective algorithm for the solution of steady-heat-conduction boundary problems for the various shells (additionally weakened by holes and cuts of various configurations) by the schemes described, for example, in [9].

Closed Toroidal Shell

Suppose that the median surface of the toroidal shell (Fig. 1) is described by the equations

$$x = (R + a\sin\varphi)\cos\nu;$$

$$y = (R + a\sin\varphi)\sin\nu; \quad z = a\cos\varphi;$$

$$0 \leqslant \nu \leqslant 2\pi; \quad -\pi \leqslant \varphi \leqslant \pi; \quad R > a.$$

Then, if the case of thermal insulation of the external shell surfaces is considered, the homogeneous equation of steady heat conduction for the shell is [10]

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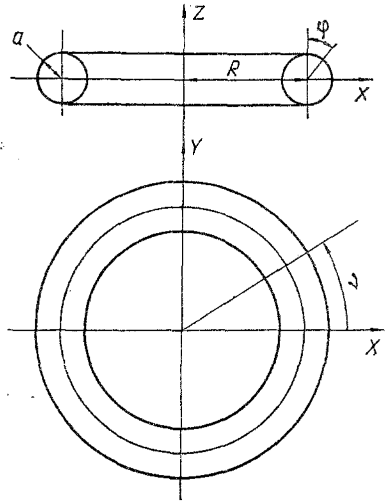

Fig. 1. Parametrization of median surface of toroidal shell.

$$\frac{\partial^2 T}{\partial \varphi^2} + \frac{a \cos\varphi}{R + a \sin\varphi} \cdot \frac{\partial T}{\partial \varphi} + \frac{a^2}{(R + a\sin\varphi)^2} \cdot \frac{\partial^2 T}{\partial \nu^2} = 0. \tag{1}$$

The temperature $T = T(\varphi, \nu)$ is written in the form of an expansion,

$$T(\varphi, \nu) = \frac{1}{\pi} \sum_{m=0}^{\infty} T_m(\varphi) \cos m(\nu - \nu_0), \tag{2}$$

which leads to the following ordinary differential equations for the coefficients $T_m(\varphi)$:

$$T_m''(\varphi) + \frac{a\cos\varphi}{R + a\sin\varphi} T_m'(\varphi) - \frac{a^2 m^2}{(R + a\sin\varphi)^2} T_m(\varphi) = 0. \tag{3}$$

It is simple to verify that

$$\begin{gathered}
\Phi_0^1(\varphi) = \text{const},\ T_0^2(\varphi) = \frac{2a}{\sqrt{R^2 - a^2}} \operatorname{arctg} \frac{R \operatorname{tg}\frac{\varphi}{2} + a}{\sqrt{R^2 - a^2}}, \\
\Phi_m^1(\varphi) = \exp\left[\frac{2am}{\sqrt{R^2 - a^2}} \operatorname{arctg} \frac{R \operatorname{tg}\frac{\varphi}{2} + a}{\sqrt{R^2 - a^2}}\right], \\
\Phi_m^2(\varphi) = \exp\left[-\frac{2am}{\sqrt{R^2 - a^2}} \operatorname{arctg} \frac{R \operatorname{tg}\frac{\varphi}{2} + a}{\sqrt{R^2 + a^2}}\right]
\end{gathered} \tag{4}$$

are linearly independent particular solutions of Eq. (3); i.e., they form a fundamental system which may be used, together with the well-known algorithm of [1], to construct Green's functions of the operator in Eq. (3). For example, for a shell that is closed in the direction φ, these functions are

$$g_m(\varphi, \varphi_0) = \begin{cases} -\dfrac{1}{2m} \exp\left[\dfrac{2am}{\sqrt{R^2 - a^2}} (\Psi(\varphi) - \Psi(\varphi_0))\right], & \varphi \leqslant \varphi_0, \\[2ex] -\dfrac{1}{2m} \exp\left[-\dfrac{2am}{\sqrt{R^2 - a^2}} (\Psi(\varphi) - \Psi(\varphi_0))\right], & \varphi \geqslant \varphi_0, \end{cases}$$

$$g_0(\varphi, \varphi_0) = \begin{cases} -\dfrac{a}{2\sqrt{R^2 - a^2}} (\Psi(\varphi) - \Psi(\varphi_0)), & \varphi \leqslant \varphi_0, \\[2ex] -\dfrac{a}{2\sqrt{R^2 - a^2}} (\Psi(\varphi_0) - \Psi(\varphi)), & \varphi \geqslant \varphi_0, \end{cases}$$

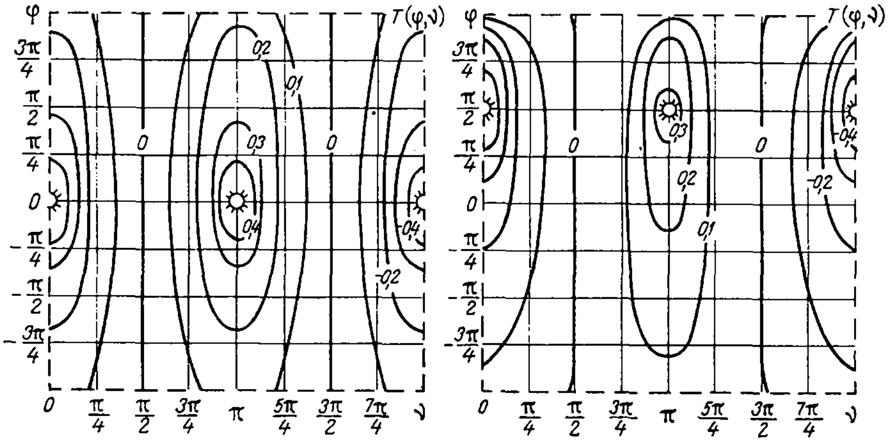

Fig. 2. Temperature field in toroidal shell under the action of point sources and sinks.

where

$$\Psi(\varphi) = \operatorname{arctg} \frac{R \operatorname{tg} \frac{\varphi}{2} + a}{\sqrt{R^2 - a^2}}.$$

Taking into account Eq. (2), the following expression is obtained for Green's function in the partial derivatives of Eq. (1):

$$G(\varphi, \nu; \varphi_0, \nu_0) = \sum_{m=0}^{\infty} g_m(\varphi, \varphi_0) \cos m(\nu - \nu_0).$$

Summing this series gives the final representation for Green's function:

$$G(\varphi, \nu; \varphi_0, \nu_0) = -\frac{1}{4\pi} \ln \left[\operatorname{ch} \frac{2a}{\sqrt{R^2 - a^2}} (\Psi(\varphi) - \Psi(\varphi_0)) - \cos(\nu - \nu_0) \right]. \quad (5)$$

It is readily established that Eq. (5) satisfies all the defining properties of a Green function.

The dependence of the function in Eq. (5) on φ and ν is shown in Fig. 2 for two different values of the coordinates φ_0, ν_0 of the heat source. A striking feature of Fig. 2 is the difference in localization of the heat from the sources (cooling from the sinks); this is a natural consequence of the change in sign of the Gaussian curvature of the median surface of the shell, which gives effectively nonequilibrium regions of positive and negative curvature.

If the fraction of the circumference in the direction parallel to ν = const is 1/p, Green's function is written in the form

$$G^p(\varphi, \nu; \varphi_0, \nu_0) = -\frac{1}{4\pi} \ln \left[\operatorname{ch} \frac{2ap}{\sqrt{R^2 - a^2}} (\Psi(\varphi) - \Psi(\varphi_0)) - \cos p(\nu - \nu_0) \right]; \quad (6)$$

when p = 1, Eq. (6) coincides with Eq. (5).

Closed Spherical Shell and Segment

The parametrization adopted for the median surface of a spherical shell of radius R (Fig. 3) is as follows

$$x = R \sin\varphi \cos\nu; \quad y = R \sin\varphi \sin\nu; \quad z = R \cos\varphi;$$
$$0 \leqslant \nu \leqslant 2\pi; \quad 0 \leqslant \varphi \leqslant \pi.$$

The steady-heat-conduction operator for this shell takes the form

$$\frac{\partial^2}{\partial\varphi^2} + \operatorname{ctg}\varphi \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi} \cdot \frac{\partial^2}{\partial\nu^2}.$$

Calculations by the scheme used above for a toroidal shell lead to the following expression for Green's function of a closed spherical shell:

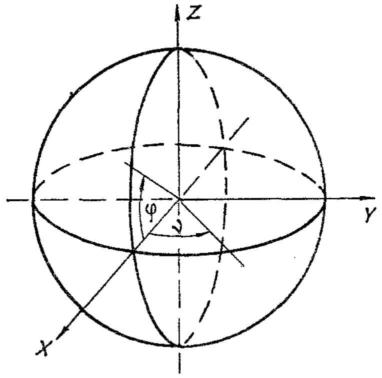

Fig. 3. Parametrization of median surface of a spherical shell.

$$G(\varphi, \nu; \varphi_0, \nu_0) = -\frac{1}{4\pi} \ln\left[\mathrm{tg}^2 \frac{\varphi}{2} - 2\,\mathrm{tg}\frac{\varphi}{2}\,\mathrm{tg}\frac{\varphi_0}{2}\cos(\nu - \nu_0) + \mathrm{tg}^2\frac{\varphi_0}{2}\right].$$

For a spherical segment of aperture angle φ_1, with boundary conditions of the first kind at the edge of the segment, Green's function is

$$G(\varphi, \nu; \varphi_0, \nu_0) = -\frac{1}{4\pi} \ln\left[\frac{\mathrm{tg}^2\frac{\varphi}{2} - 2\,\mathrm{tg}\frac{\varphi}{2}\,\mathrm{tg}\frac{\varphi_0}{2}\cos(\nu - \nu_0) + \mathrm{tg}^2\frac{\varphi_0}{2}}{\mathrm{tg}^2\frac{\varphi_1}{2} - 2\,\mathrm{tg}\frac{\varphi}{2}\,\mathrm{tg}\frac{\varphi_0}{2}\cos(\nu - \nu_0) + \dfrac{\mathrm{tg}^2\frac{\varphi}{2}\,\mathrm{tg}^2\frac{\varphi_0}{2}}{\mathrm{tg}^2\frac{\varphi_1}{2}}}\right].$$

If the fraction of the circumference in the direction ν = const is $1/p$, Green's function for a closed shell and a spherical segment are, respectively, as follows:

$$G(\varphi, \nu; \varphi_0, \nu_0) = -\frac{1}{4\pi} \ln\left[\mathrm{tg}^{2p}\frac{\varphi}{2} - 2\,\mathrm{tg}^p\frac{\varphi}{2}\,\mathrm{tg}^p\frac{\varphi_0}{2}\cos p(\nu - \nu_0) + \mathrm{tg}^{2p}\frac{\varphi_0}{2}\right];$$

$$G(\varphi, \nu; \varphi_0, \nu_0) = -\frac{1}{4\pi} \ln\left[\frac{\mathrm{tg}^{2p}\frac{\varphi}{2} - 2\,\mathrm{tg}^p\frac{\varphi}{2}\,\mathrm{tg}^p\frac{\varphi_0}{2}\cos p(\nu - \nu_0) + \mathrm{tg}^{2p}\frac{\varphi_0}{2}}{\mathrm{tg}^{2p}\frac{\varphi_1}{2} - 2\,\mathrm{tg}^p\frac{\varphi}{2}\,\mathrm{tg}^p\frac{\varphi_0}{2}\cos p(\nu - \nu_0) + \dfrac{\mathrm{tg}^{2p}\frac{\varphi}{2}\,\mathrm{tg}^{2p}\frac{\varphi_0}{2}}{\mathrm{tg}^{2p}\frac{\varphi_1}{2}}}\right].$$

If the expressions obtained for a Green function of the steady-heat-conduction operator for spherical and toroidal shells are used as the core of a potential representation (see [1, 9, 10], for example), temperature fields may be determined in thin shell structures of complex form — for example, shells with holes and cuts of different configurations. They may also be used to calculate the temperature in such geometrically complex bodies in the presence of sources of external thermal energy distributed according to an arbitrary law.

NOTATION

φ, ν, curvilinear mutually orthogonal coordinates on the median surface; a, radius of the meridional section of the shell; R, distance from the center of the meridional section to the axis of rotation; $G(\varphi, \nu; \varphi_0, \nu_0)$, Green's function.

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